Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.5, 2016

Bayesian Analysis for Generalized Rayleigh Distribution

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Abstract

The generalized Rayleigh distribution (GRD) is considered to be a very useful life distribution. In this paper, we obtain Bayesian estimation of the shape parameter of the two-parameter Generalized Rayleigh distribution using single and double priors. A simulation study is conducted in R software to compare the different priors.

Key Words: Bayes estimation, double prior, hyper parameter, posterior distribution, posterior predictive distribution.

1. Introduction

The Rayleigh distribution is one of the most popular distributions in analyzing skewed data. The Rayleigh distribution was originally proposed in the fields of acoustics and optics by Lord Rayleigh (1880), and it became widely known since then in oceanography, and in communication theory for describing instantaneous peak power of received radio signals. It has found use in engineering and physics for modeling wave propagation, radiation, synthetic aperture radar images, and other related phenomena. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava (1993). It presents a fexible family in the varieties of shapes and is suitable for modeling data with different types of hazard rate function. The two-parameter Burr-Type X distribution, also called generalized Rayleigh distribution, has been studied by various authors, see Surles and Padjett (1998), Jaheen (1995, 1996), Sartawi and Abu-Salih (1991), Ahmad et al. (1997), Raqab (2006), Aslam (2008). Recently Surles and Padjett (2001, 2005) introduced a two parameter Burr type X distributions that can be used quite effectively in modeling strength data and also modeling general lifetime data. Kundu and Raqab (2005) considered different estimators by comparing the maximum likelihood estimators, the modified moment estimators and estimates based on percentiles using simulation techniques. Reshi et. al. (2013) obtained Bayes estimators of size biased Generalized Rayleigh distribution under extension of Jeffery's prior.

The generalized Rayleigh distribution (GRD) has the pdf of the form

$$f(x\,|\,\alpha,\sigma)=\frac{2\alpha}{\sigma^2}xe^{-\left(\frac{x}{\sigma}\right)^2}\left(1-e^{-\left(\frac{x}{\sigma}\right)^2}\right)^{\alpha-1}\qquad ;x>0;\alpha>0,\sigma>0 \tag{1}$$

where α is the shape parameter and σ is the scale parameter of this distribution.

For $\alpha=1$, $f(x\,|\,\sigma)=\dfrac{2}{\sigma^2}xe^{-\left(\frac{x}{\sigma}\right)^2}$ $;x>0,\ \sigma>0$, we get the standard Rayleigh distribution.

The cdf of GRD is given by

$$F(x \mid \alpha, \sigma) = \int_0^{1-e^{-\left(\frac{x}{\sigma}\right)^2}} \alpha\, y^{\alpha-1}\, dy = \left(1 - e^{-\left(\frac{x}{\sigma}\right)^2}\right)^{\alpha}$$

2. Maximum Likelihood Estimation

Let $X_1, X_2 \ldots X_n$ be independently and identically distributed random variables from GRD (α, σ) defined in (1), then the likelihood equation of α is given by

$$L(\alpha \mid x, \sigma) = \left(\frac{2\alpha}{\sigma^2}\right)^n \prod_{i=1}^n x_i\, e^{-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^2} \prod_{i=1}^n \left(1 - e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)^{\alpha-1} \quad (2)$$

$$\ln L = n \ln 2 + n \ln \alpha - 2n \ln \sigma + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^2 + (\alpha - 1) \sum_{i=1}^n \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$$

The maximum likelihood estimate of α is

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^n \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)} = \frac{n}{T}, \qquad \text{where } T = \sum_{i=1}^n \ln\left(1 - e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)^{-1} \quad (3)$$

3. Bayesian Estimation Using Different Priors

In this section, we present different single priors, viz., exponential prior, gamma prior, chi-square prior and a non-informative (extension of Jeffrey's prior) and double priors like exponential-gamma prior, gamma-chi-square prior, and chi-square-exponential prior.

The likelihood equation of α, keeping σ constant, is given by

$$L(\alpha \mid x, \sigma) = \left(\frac{2\alpha}{\sigma^2}\right)^n \prod_{i=1}^n x_i\, e^{-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^2} \prod_{i=1}^n \left(1 - e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)^{\alpha-1}$$

$$\propto \alpha^n \exp\left(\sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^2\right) e^{-(\alpha-1)T}, \qquad T = \sum_{i=1}^n \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right) \quad (4)$$

3.1 Bayesian estimation using Gamma prior:

Assume that α has a Gamma prior with hyper parameters $(a_1, b_1) > 0$ defined by

$$g(\alpha) = \frac{a_1^{b_1}}{\Gamma(b_1)} e^{-a_1 \alpha} \alpha^{b_1 - 1} \qquad ; \alpha > 0; (a_1, b_1) > 0 \quad (5)$$

Using Bayes theorem the posterior distribution of α is defined by

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.5, 2016

$$\pi_1(\alpha \mid x) \propto L(\alpha \mid x) g(\alpha)$$

$$\propto \alpha^n \exp\left(\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2\right) e^{-(\alpha-1)T} e^{-a_1 \alpha} \alpha^{b_1-1}$$

$$\pi_1(\alpha \mid x) = K \exp\left(\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2 + T\right) e^{-(T+a_1)\alpha} \alpha^{b_1+n-1} \tag{6}$$

where K is obtained by the relation:

$$\int_0^\infty \pi_1(\alpha \mid x) d\alpha = 1$$

$$K = \frac{(T+a_1)^{b_1+n}}{\exp\left(\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2 + T\right)\Gamma(b_1+n)}$$

Using value of K in (6) the posterior distribution of α is given by

$$\pi_1(\alpha \mid x) = \frac{(T+a_1)^{b_1+n}}{\Gamma(b_1+n)} e^{-(T+a_1)\alpha} \alpha^{b_1+n-1} \tag{7}$$

which is a gamma distribution with parameters $\theta_1 = (T + a_1), \beta_1 = (b_1 + n)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_1, \beta_1)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{b_1+n}{T+a_1} = \frac{\beta_1}{\theta_1}$$

3.2 Bayesian estimation using Chi-square prior:

Assume that α has a Chi-square prior with hyper parameter $c_1 > 0$ defined by

$$g(\alpha) = \frac{1}{2^{\frac{c_1}{2}}\Gamma\left(\frac{c_1}{2}\right)} e^{-\frac{\alpha}{2}} \alpha^{\frac{c_1}{2}-1} \qquad ; \alpha > 0, c_1 > 0 \tag{8}$$

The posterior distribution of α is defined by

$$\pi_2(\alpha \mid x) = \frac{\left(T+\frac{1}{2}\right)^{b_1+n}}{\Gamma\left(\frac{c_1}{2}+n\right)} e^{-\left(T+\frac{1}{2}\right)\alpha} \alpha^{\frac{c_2}{2}+n-1} \tag{9}$$

which is a gamma distribution with parameters $\theta_2 = \left(T + \frac{1}{2}\right), \beta_2 = \left(\frac{c_1}{2} + n\right)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_2, \beta_2)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{\frac{c_1}{2} + n}{T + \frac{1}{2}} = \frac{\beta_2}{\theta_2}$$

3.3 Bayesian estimation using Exponential prior:

Assume that α has an exponential prior with hyper parameter $d_1 > 0$ defined by

$$g(\alpha) = \frac{1}{d_1} e^{-\frac{\alpha}{d_1}} \quad ;\alpha > 0; d_1 > 0 \tag{10}$$

The posterior distribution of α is given by

$$\pi_3(\alpha \mid x) = \frac{\left(T + \frac{1}{c_1}\right)^{n+1}}{\Gamma(n+1)} e^{-\left(T + \frac{1}{d_1}\right)\alpha} \alpha^n \tag{11}$$

which is a gamma distribution with parameters $\theta_3 = \left(T + \frac{1}{d_1}\right), \beta_3 = (n+1)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_3, \beta_3)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{n+1}{T + \frac{1}{d_1}} = \frac{\beta_3}{\theta_3}$$

3.4 Bayesian estimation using extension of Jeffreys' prior:

Assume that α has an extension of Jeffreys' prior defined by

$$g(\alpha) = [I(\alpha)]^m \quad ;\alpha > 0; m \in R^+$$

where $I(\alpha) = -E\left[\frac{\partial^2}{\partial \alpha^2} \ln L(\alpha \mid x)\right] = \frac{n}{\alpha^2}$

Thus, the extension of Jeffreys prior is given by:

$$g(\alpha) \propto \frac{1}{\alpha^{2m}}, \alpha > 0 \tag{12}$$

The posterior distribution of α is given by

$$\pi_4(\alpha \mid x) = \frac{T^{n-2m+1}}{\Gamma(n-2m+1)} e^{-\alpha T} \alpha^{n-2m} \tag{13}$$

which is a gamma distribution with parameters $\theta_4 = T, \beta_4 = (n-2m+1)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_4, \beta_4)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{n-2m+1}{T} = \frac{\beta_4}{\theta_4}$$

Remark: For $m = \frac{1}{2}$ (Jeffreys' prior), the Bayes estimate becomes $\frac{n}{T}$ which implies that the Bayes estimate under Jeffreys' prior is same as the MLE.

3.5 Bayesian estimation using Gamma-Chi-square prior:

Assume that α has a Gamma prior with hyper parameters $(a_2, b_2)>0$ defined by

$$g_1(\alpha) = \frac{a_2^{\,b_2}}{\Gamma(b_2)} e^{-a_2 \alpha} \alpha^{b_2 - 1} \quad ; \alpha > 0; (a_2, b_2) > 0 \tag{14}$$

Again assume that α has a Chi-square prior with hyper parameter $c_2 > 0$ defined by

$$g_2(\alpha) = \frac{1}{2^{\frac{c_2}{2}} \Gamma\left(\frac{c_2}{2}\right)} e^{-\frac{\alpha}{2}} \alpha^{\frac{c_2}{2} - 1} \qquad ; \alpha > 0, c_2 > 0 \tag{15}$$

So that the double is given by

$$g_{11} \propto g_1(\alpha) g_2(\alpha)$$

$$g_{11} \propto e^{-\left(a_2 + \frac{1}{2}\right)\alpha} \alpha^{\frac{c_2}{2} + b_2 - 2} \tag{16}$$

Therefore, the posterior distribution of α becomes

$$\pi_5(\alpha \mid x) = \frac{\left(T + a_2 + \frac{1}{2}\right)^{\frac{c_2}{2} + b_2 + n - 1}}{\Gamma\left(\frac{c_2}{2} + b_2 + n - 1\right)} e^{-\left(T + a_2 + \frac{1}{2}\right)\alpha} \alpha^{\frac{c_2}{2} + b_2 + n - 2} \tag{17}$$

which is a gamma distribution with parameters $\theta_5 = \left(T + a_2 + \frac{1}{2}\right), \beta_5 = \left(\frac{c_2}{2} + b_2 + n - 1\right)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_5, \beta_5)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{\frac{c_2}{2} + b_2 + n - 1}{T + a_2 + \frac{1}{2}} = \frac{\beta_5}{\theta_5}$$

3.6 Bayesian estimation using Chi-square-Exponential prior distribution:

Assume that α has a Chi-square prior with hyper parameter $c_3 > 0$ defined by

$$g_3(\alpha) = \frac{1}{2^{\frac{c_3}{2}} \Gamma\left(\frac{c_3}{2}\right)} e^{-\frac{\alpha}{2}} \alpha^{\frac{c_3}{2} - 1} \qquad ; \alpha > 0, c_3 > 0$$

Again assume that α has an exponential prior with hyper parameter $d_2 > 0$ defined by

$$g_4(\alpha) = \frac{1}{d_2} e^{-\frac{\alpha}{d_2}} \quad ; \alpha > 0; d_2 > 0$$

So that the double is given by

$$g_{22} \propto g_3(\alpha) g_4(\alpha)$$

$$g_{22} \propto e^{-\left(\frac{1}{d_2} + \frac{1}{2}\right)\alpha} \alpha^{\frac{c_3}{2} - 1} \tag{18}$$

Therefore, the posterior distribution of α becomes

$$\pi_6(\alpha \mid x) = \frac{\left(T + \frac{1}{d_2} + \frac{1}{2}\right)^{\frac{c_3}{2} + n}}{\Gamma\left(\frac{c_3}{2} + n\right)} e^{-\left(T + \frac{1}{d_2} + \frac{1}{2}\right)\alpha} \alpha^{\frac{c_3}{2} + n - 1} \tag{19}$$

which is a gamma distribution with parameters $\theta_6 = \left(T + \frac{1}{d_2} + \frac{1}{2}\right), \beta_6 = \left(\frac{c_3}{2} + n\right)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$, i.e., $\alpha \mid x \sim G(\theta_6, \beta_6)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{\frac{c_3}{2} + n}{T + \frac{1}{d_2} + \frac{1}{2}} = \frac{\beta_6}{\theta_6}$$

3.7 Bayesian estimation using Exponential-Gamma prior:

Again assume that α has an exponential prior with hyper parameter $d_3 > 0$ defined by

$$g_5(\alpha) = \frac{1}{d_3} e^{-\frac{\alpha}{d_3}} \quad ;\alpha > 0; d_3 > 0$$

Assume that α has a Gamma prior with hyper parameters (a_3, b_3)>0 defined by

$$g_6(\alpha) = \frac{a_3^{\,b_3}}{\Gamma(b_3)} e^{-a_3 \alpha} \alpha^{b_3 - 1} \; ;\alpha > 0,\ (a_3, b_3) > 0$$

So that the double is given by

$$g_{33} \propto g_5(\alpha) g_6(\alpha)$$

$$g_{33} \propto e^{-\left(a_3 + \frac{1}{d_3}\right)\alpha} \alpha^{b_3 - 1} \tag{20}$$

Therefore, the posterior distribution of α becomes

$$\pi_7(\alpha \mid x) = \frac{\left(T + a_3 + \frac{1}{d_3}\right)^{b_3 + n}}{\Gamma(b_3 + n)} e^{-\left(T + a_3 + \frac{1}{2}\right)\alpha} \alpha^{b_3 + n - 1} \tag{21}$$

which is a gamma distribution with parameters $\theta_7 = \left(T + a_3 + \frac{1}{d_3}\right), \beta_7 = (b_3 + n)$, where $T = \sum_{i=1}^{n} \ln\left(1 = e^{-\left(\frac{x_i}{\sigma}\right)^2}\right)$

, i.e., $\alpha \mid x \sim G(\theta_7, \beta_7)$.

The Bayes estimate of $\alpha \mid x$ is given by

$$E(\alpha \mid x) = \frac{b_3 + n}{T + a_3 + \frac{1}{d_3}} = \frac{\beta_7}{\theta_7}$$

4. Posterior Variances Under Different Assumed Priors:

The variances of the posterior distribution under all of assumed informative priors are calculated by assuming different set of values for hyper parameters, different sample size and different value of parameter which is given by

$$V(\alpha \mid x) = \frac{\beta_i}{\theta_i^{\,2}}, i = 1, \ldots 7 \tag{22}$$

where θ_i and β_i are shape and rate parameters of gamma distribution.

5. Posterior predictive distribution under various priors

After we have observed a sample $X_1, X_2 \ldots X_n$ from the population, the relevant predictive distribution for a new observation is called a posterior predictive distribution, because it conditions on an observed dataset. In the case of a GRD with different prior distributions, we obtain the posterior predictive distribution as follows:

5.1 Posterior predictive distribution under Gamma prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2 .. x_n)$ under the gamma prior is defined by

$$\phi_1(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_1(\alpha\,|\,x)d\alpha = \int_0^\infty \frac{2\alpha}{\sigma^2} y e^{-\left(\frac{y}{\sigma}\right)^2}\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{\alpha-1} \frac{\theta_1^{\beta_1}}{\Gamma(\beta_1)} e^{-\theta_1\alpha}\alpha^{\beta_1-1}d\alpha$$

$$= \int_0^\infty \frac{2\alpha}{\sigma^2} y\left(1-e^{-s}\right)e^{-(\alpha-1)s} \frac{\theta_1^{\beta_1}}{\Gamma(\beta_1)} e^{-\theta_1\alpha}\alpha^{\beta_1-1}d\alpha \text{ , where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

$$= \frac{\theta_1^{\beta_1}}{\Gamma(\beta_1)} \frac{2y}{\sigma^2}\left(1-e^{-s}\right)e^{s}\int_0^\infty e^{-(\theta_1+s)\alpha}\,\alpha^{\beta_1}d\alpha$$

$$= \frac{2\beta_1}{\sigma^2} \frac{\frac{y}{\theta_1}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_1}\right)^{\beta_1+1}} \text{ , where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_1 and θ_1 is defined in (7).

5.2 Posterior predictive distribution under Chi-square prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2 .. x_n)$ under the chi-square prior is defined by

$$\phi_2(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_2(\alpha\,|\,x)d\alpha = \frac{2\beta_2}{\sigma^2} \frac{\frac{y}{\theta_2}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_2}\right)^{\beta_2+1}} \text{ , where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_2 and θ_2 is defined in (9).

5.3 Posterior predictive distribution under Exponential prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2 .. x_n)$ under the exponential prior is defined by

$$\phi_3(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_3(\alpha\,|\,x)d\alpha = \frac{2\beta_3}{\sigma^2} \frac{\frac{y}{\theta_3}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_3}\right)^{\beta_1+1}} \text{ , where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_3 and θ_3 is defined in (11).

5.4 Posterior predictive distribution under extension of Jeffreys' prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2..x_n)$ under the extension of Jeffreys' prior is defined by

$$\phi_4(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_4(\alpha\,|\,x)d\alpha = \frac{2\beta_4}{\sigma^2}\frac{\frac{y}{\theta_4}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_4}\right)^{\beta_4+1}}, \text{ where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_4 and θ_4 is defined in (13).

5.5 Posterior predictive distribution under Gamma-Chi-square prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2..x_n)$ under the gamma-chi-square prior is defined by

$$\phi_5(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_5(\alpha\,|\,x)d\alpha = \frac{2\beta_5}{\sigma^2}\frac{\frac{y}{\theta_5}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_5}\right)^{\beta_5+1}}, \text{ where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_5 and θ_5 is defined in (17).

5.6 Posterior predictive distribution under Chi-square-Exponential prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2..x_n)$ under the chi-square-exponential prior is defined by

$$\phi_6(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_6(\alpha\,|\,x)d\alpha = \frac{2\beta_6}{\sigma^2}\frac{\frac{y}{\theta_6}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_6}\right)^{\beta_6+1}}, \text{ where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_6 and θ_6 is defined in (19).

5.7 Posterior predictive distribution under Exponential-Gamma prior:

The posterior predictive distribution for $y = x_{n+1}$ given $\underline{x} = (x_1, x_2..x_n)$ under the exponential-gamma prior is defined by

$$\phi_7(y\,|\,x) = \int_0^\infty f(y\,|\,\alpha)\pi_7(\alpha\,|\,x)d\alpha = \frac{2\beta_7}{\sigma^2}\frac{\frac{y}{\theta_7}\left(e^s-1\right)}{\left(1+\frac{s}{\theta_7}\right)^{\beta_7+1}}, \text{ where } s = \ln\left(1-e^{-\left(\frac{y}{\sigma}\right)^2}\right)^{-1}$$

where β_7 and θ_7 is defined in (21).

6. Simulation:

In our simulation study, we chose different samples of size of 25, 50 and 100 to represent small, medium and large data set. The shape parameter is estimated for generalized Rayleigh distribution by using Bayesian method of estimation under various types of priors. The simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes by using various types of priors. The results are presented in tables from table 1 to 3 given below:

Table 1: Posterior Mean and SE for the posterior distribution using different priors with n=25.

α	Hyper parameter $a_i=b_i=c_i=d_i$	Posterior mean/SE	Gamma prior	Chi-square prior	Exponential prior	Extension of Jeffrey's prior		Gamma-Chi-square prior	Chi-square-Exponential prior	Exponential-Gamma prior
						m=0.5	m=1.0			
1	0.5	Mean	1.3492	1.3360	1.2745	1.3587	1.3043	1.2758	1.2081	1.2201
		SE	0.2672	0.2659	0.2499	0.2717	0.2662	0.2564	0.2404	0.2416
	1.0	Mean	1.3402	1.3492	1.3402	1.3587	1.3043	1.2814	1.2814	1.2745
		SE	0.2628	0.2672	0.2628	0.2717	0.2662	0.2538	0.2538	0.2499
	2.0	Mean	1.3235	1.3756	1.3756	1.3587	1.3043	1.2919	1.3402	1.2919
		SE	0.2547	0.2698	0.2698	0.2717	0.2662	0.2486	0.2628	0.2486
2	0.5	Mean	5.0442	4.9948	3.9663	5.4881	5.2686	4.4552	3.5789	3.6143
		SE	0.9989	0.9940	0.7778	1.0976	1.0755	0.8955	0.7122	0.7157
	1.0	Mean	4.6802	5.0442	4.6802	5.4881	5.2686	4.2112	4.2112	3.9663
		SE	0.9179	0.9989	0.9179	1.0976	1.0755	0.8339	0.8339	0.7778
	2.0	Mean	4.1188	5.1431	5.1431	5.4881	5.2686	3.8269	4.6802	3.8269
		SE	0.7927	1.0087	1.0087	1.0976	1.0755	0.7365	0.9179	0.7365
5	0.5	Mean	11.8830	11.7665	7.1313	15.1891	14.5815	9.3540	6.0903	6.1506
		SE	2.3532	2.3416	1.3986	3.0378	2.9764	1.8802	1.2120	1.2180
	1.0	Mean	9.8265	11.8830	9.8265	15.1891	14.5815	8.1057	8.1057	7.1313
		SE	1.9271	2.3532	1.9271	3.0378	2.9764	1.6052	1.6052	1.3986
	2.0	Mean	7.4055	12.1160	12.1160	15.1891	14.5815	6.5124	9.8265	6.5124
		SE	1.4252	2.3761	2.3761	3.0378	2.9764	1.2533	1.9271	1.2533

Table 2: Posterior Mean and SE for the posterior distribution using different priors with n=50.

α	Hyper parameter $a_i=b_i=c_i=d_i$	Posterior mean/SE	Gamma prior	Chi-square prior	Exponential prior	Extension of Jeffrey's prior		Gamma-Chi-square prior	Chi-square-Exponential prior	Exponential-Gamma prior
						m=0.5	m=1.0			
1	0.5	Mean	1.0057	1.0008	0.9862	1.0058	0.9857	0.9810	0.9624	0.9672
		SE	0.1415	0.1412	0.1381	0.1422	0.1408	0.1391	0.1358	0.1361
	1.0	Mean	1.0057	1.0057	1.0057	1.0058	0.9857	0.9861	0.9861	0.9862
		SE	0.1408	0.1415	0.1408	0.1422	0.1408	0.1388	0.1388	0.1381
	2.0	Mean	1.0056	1.0157	1.0157	1.0058	0.9857	0.9960	1.0057	0.9960
		SE	0.1394	0.1422	0.1422	0.1422	0.1408	0.1381	0.1408	0.1381
2	0.5	Mean	2.2861	2.2748	2.1619	2.3159	2.2696	2.2023	2.0859	2.0963
		SE	0.3217	0.3209	0.3027	0.3275	0.3242	0.3122	0.2943	0.2950
	1.0	Mean	2.2576	2.2861	2.2576	2.3159	2.2696	2.1871	2.1871	2.1619
		SE	0.3161	0.3217	0.3161	0.3275	0.3242	0.3078	0.3078	0.3027
	2.0	Mean	2.2043	2.3087	2.3087	2.3159	2.2696	2.1586	2.2576	2.1586
		SE	0.3057	0.3233	0.3233	0.3275	0.3242	0.2993	0.3161	0.2993
5	0.5	Mean	52.1971	51.9387	20.6688	106.9547	104.8156	33.9015	16.9335	17.0178
		SE	7.3451	7.3269	2.8942	15.1257	14.9737	4.8064	2.3888	2.3947
	1.0	Mean	34.7533	52.1971	34.7533	106.9547	104.8156	25.6673	25.6673	20.6688
		SE	4.8664	7.3451	4.8664	15.1257	14.9737	3.6119	3.6119	2.8942
	2.0	Mean	21.0741	52.7139	52.7139	106.9547	104.8156	17.5232	34.7533	17.5232
		SE	2.9224	7.3814	7.3814	15.1257	14.9737	2.4300	4.8664	2.4300

Table 3: Posterior Mean and SE for the posterior distribution using different priors with n=100.

α	Hyper parameters $a_i=b_i=c_i=d_i$	Posterior mean/SE	Gamma prior	Chi-square prior	Exponential prior	Extension of Jeffrey's prior		Gamma-Chi-square prior	Chi-square-Exponential prior	Exponential-Gamma prior
						m=0.5	m=1.0			
1	0.5	Mean	1.0436	1.0410	1.0327	1.0438	1.0334	1.0304	1.0198	1.0223
		SE	0.1041	0.1040	0.1028	0.1044	0.1039	0.1032	0.1019	0.1020
	1.0	Mean	1.0433	1.0436	1.0433	1.0438	1.0334	1.0328	1.0328	1.0327
		SE	0.1038	0.1041	0.1038	0.1044	0.1039	0.1030	0.1030	0.1028
	2.0	Mean	1.0429	1.0488	1.0488	1.0438	1.0334	1.0376	1.0433	1.0376
		SE	0.1033	0.1044	0.1044	0.1044	0.1039	0.1027	0.1038	0.1027
2	0.5	Mean	2.7388	2.7320	2.6443	2.7628	2.7352	2.6818	2.5908	2.5972
		SE	0.2732	0.2729	0.2631	0.2763	0.2749	0.2685	0.2588	0.2591
	1.0	Mean	2.7154	2.7388	2.7154	2.7628	2.7352	2.6661	2.6661	2.6443
		SE	0.2702	0.2732	0.2702	0.2763	0.2749	0.2660	0.2660	0.2631
	2.0	Mean	2.6705	2.7524	2.7524	2.7628	2.7352	2.6360	2.7154	2.6360
		SE	0.2644	0.2739	0.2739	0.2763	0.2749	0.2610	0.2702	0.2610
5	0.5	Mean	9.8139	9.7895	8.6027	10.2664	10.1637	9.2873	8.1900	8.2104
		SE	0.9790	0.9777	0.8560	1.0266	1.0215	0.9299	0.8180	0.8190
	1.0	Mean	9.4036	9.8140	9.4036	10.2664	10.1637	8.9409	8.9409	8.6027
		SE	0.9357	0.9790	0.9357	1.0266	1.0215	0.8919	0.8919	0.8560
	2.0	Mean	8.6879	9.8628	9.8628	10.2664	10.1637	8.3330	9.4036	8.3330
		SE	0.8602	0.9814	0.9814	1.0266	1.0215	0.8251	0.9357	0.8251

The posterior mean, posterior standard error under all the assumed priors is calculated by assuming the different values of hyper parameters. From table 1 to 3, it is clear that the posterior standard error under the double prior Exponential-Gamma distribution are less as compared to other assumed priors, which shows that this prior is efficient as compared to other priors.

It is also observed that the posterior standard error under the double prior Exponential-Gamma distribution and Chi-square-Exponential distribution are almost same in all the sample sizes when the value of hyper parameter is 0.5. Moreover, when the sample size increases from 25 to 100, the MSE decreases quite significantly.

Conclusion

In this paper, the problem of Bayesian estimation for the generalized Rayleigh distribution, under different priors is considered. From the results, we observe that in most cases, Bayesian Estimator under the double prior Gamma - Exponential distribution has the less posterior standard error values. It is also observed that the standard error decreases as the sample size increases from 25 to 100.

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